§ 16.3 Conservative Vector Fields –

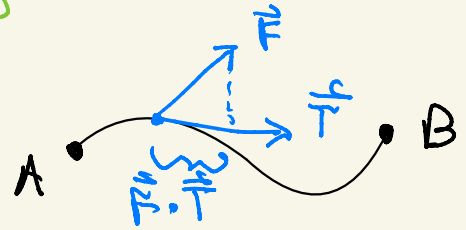
Recall: Line Integral (all equivalent)

$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_a^b \vec{F}\cdot\vec{v}\,dt = \int_C \vec{F}\cdot d\vec{r} = \int_C M\,dx + N\,dy + P\,dz$$

Tells Meaning
- Work Done
- Circulation
- "Amt of $\vec{F}$ pointing tangent to C"

tells how to compute it

"View a parameterization $\vec{r}(t)$ as a coordinate system on C"



We proved:

Thm: If $\vec{F} = \nabla f$, then $\int_C \vec{F}\cdot\vec{T}\,ds = f(B) - f(A)$

Said Differently: We can evaluate $\int_C \vec{F}\cdot\vec{T}\,ds$ if we can find an "anti-derivative" f such that $\nabla f = \vec{F}$. (f is called a "potential" – If so, we say $\vec{F}$ Conservative)

I.e., then: $\int_C \vec{F}\cdot\vec{T}\,ds = \int_C \nabla f\cdot\vec{T}\,ds = f(B) - f(A)$

Q1: Given $\vec{F}$, how do we know f exists st $\nabla f = \vec{F}$?

Q2: Given $\vec{F}$ has a potential, how do we find it?

Note: If $\vec{F}$ is conservative, so there is an f such that $\nabla f = \vec{F}$, then

indicates C is closed

$$\oint_C \vec{F}\cdot\vec{T}\,ds = 0 \qquad (f(B)-f(A)=0)$$

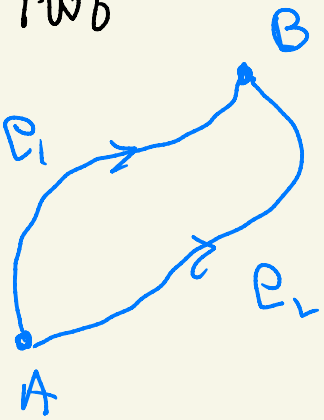
for every closed curve C.

Turns out it goes the other way:

Theorem: If $\oint_C \vec{F}\cdot\vec{T}\,ds = 0$ for every closed curve C, then $\vec{F}$ is conservative.

"**Proof:**" Assume $\oint_C \vec{F}\cdot\vec{T}\,ds = 0$ for every closed C. Then

(1) $\int_{C_1} \vec{F}\cdot\vec{T}\,ds = \int_{C_2} \vec{F}\cdot\vec{T}\,ds$ for any two curves taking $A \to B$



I.e. $C_2 - C_1$ is closed so

$$0 = \int_{C_2-C_1} \vec{F}\cdot\vec{T}\,ds = \int_{C_1} \vec{F}\cdot\vec{T}\,ds - \int_{C_2} \vec{F}\cdot\vec{T}\,ds$$

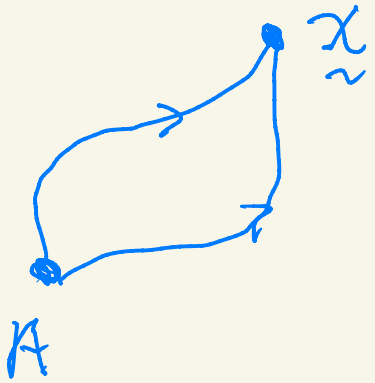
$$\Rightarrow \int_{C_1} \vec{F}\cdot\vec{T}\,ds = \int_{C_2} \vec{F}\cdot\vec{T}\,ds$$

(2) Define: $f(\underset{\sim}{x}) = \int_A^{\underset{\sim}{x}} \vec{F} \cdot \vec{T}\, ds$

This is well defined because the result is the same for every path $A \to \underset{\sim}{x}$

(So take any curve from $A \to \underset{\sim}{x} = (x, y, z)$)

(3) Turns out: $\nabla f = \vec{F}$.



(We skip this part of proof)

Problem: This is important for theoretical reason's (Complex Variable! - how to put $i = \sqrt{-1}$ into Calculus - Math 185A)

... But doesn't help us answer the question

Q1: Given $\vec{F}$, how do we know its conservative?

We now address Q1

Q1: Given $\vec{F} = (M(x,y,z), N(x,y,z), P(x,y,z))$, how do we dermine whether or not $\vec{F}$ is conservative?

There is an extraordinay answer !

To get the answer, consider following diagram:

$$\mathbb{R} \xrightarrow{\nabla} \mathbb{R}^3 \xrightarrow{\text{Curl}} \mathbb{R}^3 \xrightarrow{\text{Div}} \mathbb{R}$$

$f \uparrow \mathbb{R}^3$ $\quad \vec{F} \uparrow \mathbb{R}^3$ $\quad \vec{F} \uparrow \mathbb{R}^3$ $\quad f \uparrow \mathbb{R}^3$

The diagram Describes the fonctions which are the inputs and outputs of operators ∇, Curl, Div

Eg: $\begin{matrix}\mathbb{R}\\ \uparrow f\\ \mathbb{R}^3\end{matrix} \xrightarrow{\nabla} \nabla f = \vec{F}$ $\begin{matrix}\mathbb{R}^3\\ \uparrow\\ \mathbb{R}^3\end{matrix}$

The gradient inputs $f: \mathbb{R}^3 \to \mathbb{R}$

The gradient ouputs vector field $\nabla f = \vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$

Fact: (Not hard to check, but profoundly interesting)

Thm: If you do two in a row you get zero!

i.e. $\text{Curl}(\nabla f) = 0$ and $\text{Div}(\text{Curl}\vec{F}) = 0$

all f $\quad$ all $\vec{F}$

Example: Show $\text{Curl}(\nabla f) = 0$ for all f

Soln: $\text{Curl}(\nabla f) = \text{Curl}(f_x, f_y, f_z)$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ f_x & f_y & f_z \end{vmatrix}$$

$$= \vec{i}(\partial_y f_z - \partial_z f_y) - \vec{j}(\partial_x f_z - \partial_z f_x) + \vec{k}(\partial_x f_y - \partial_y f_x)$$

$f_{zy} - f_{yz} = 0$ $\quad$ $f_{zx} - f_{xz} = 0$ $\quad$ $f_{yx} - f_{xy} = 0$

$= 0$ ✓

Example: Show $\text{Div}(\text{Curl}\,\vec{F}) = 0$ $\forall\ \vec{F}$

(Exam Problem?)

So back to our question:

Q1: Given $\vec{F}$, how do we know its conservative, ie., how do we know $\exists f$ st $\nabla f = \vec{F}$?

We know: If $\nabla f = \vec{F}$, then $\text{Curl}\vec{F} = \text{Curl}\,\nabla f = 0$.

Does it go the other way?

I.e. is it true that if $\text{Curl}\vec{F} = 0$, then $\vec{F} = \nabla f$ some f?

If so we have a quick way of determining whether $\vec{F}$ is conservative - just take its Curl!

Ans: Yes, but only so long as $\vec{F}$ has No Singularities.

The precise theorem is -

Theorem: If $\text{Curl}\vec{F} = 0$ and $\vec{F}$ is singularity free in a simply connect Domain $D \subseteq \mathbb{R}^3$, then $\oint_C \vec{F} \cdot \vec{T}\, ds = 0$ for every closed C & $\vec{F} = \nabla f$, $f(x) = \int_A^x \vec{F} \cdot \vec{T}\, ds$

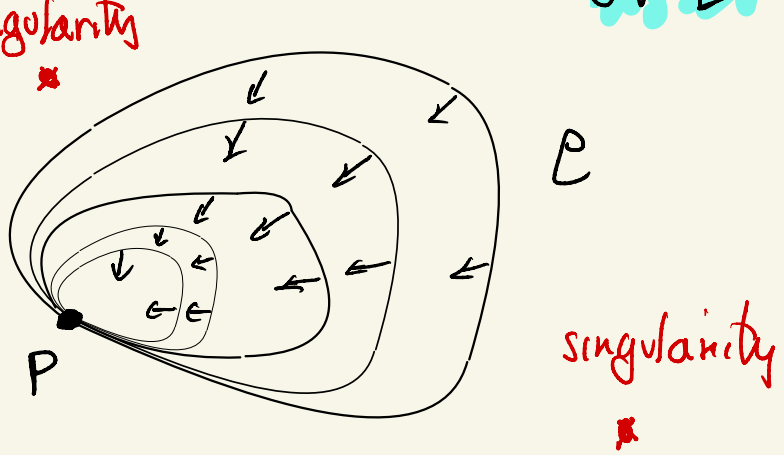
We need to define simply connected domain D

A simply connected domain is one that has

NO HOLES

Defn: $D \subseteq \mathbb{R}^3$ is simply connected if every closed curve in D can be continuously contracted to a point on C without passing out of D.

Picture: Continuously contracting C to point P



Famous Example:

$$\vec{F} = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right) \qquad \underset{\sim}{x} = (x, y)$$

$\vec{F}$ is defined for $\underset{\sim}{x} = (x,y) \neq (0,0)$.

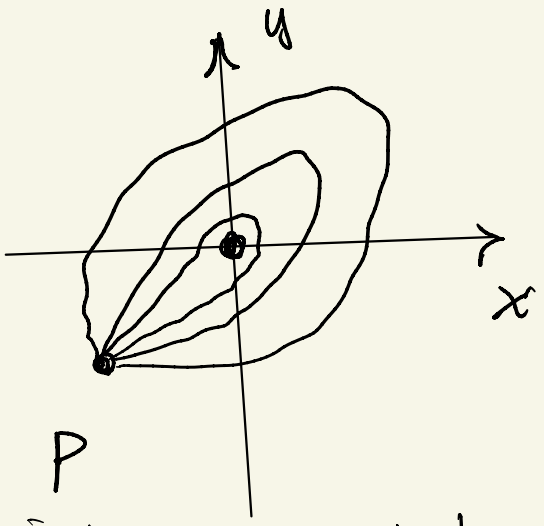
If $\mathrm{Curl}\,\vec{F} = 0$ $\underset{\sim}{x} \neq 0$ we cannot apply the theorem because $D = \{ \underset{\sim}{x} \in \mathbb{R}^2 : \underset{\sim}{x} \neq 0 \}$ is Not Simply Connected...

Because $\vec{F} = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0 \right)$ not defined $x = y = 0$,

...so D not simply connected!

Picture: $\vec{F} = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$ $\underset{\sim}{x} \neq 0$

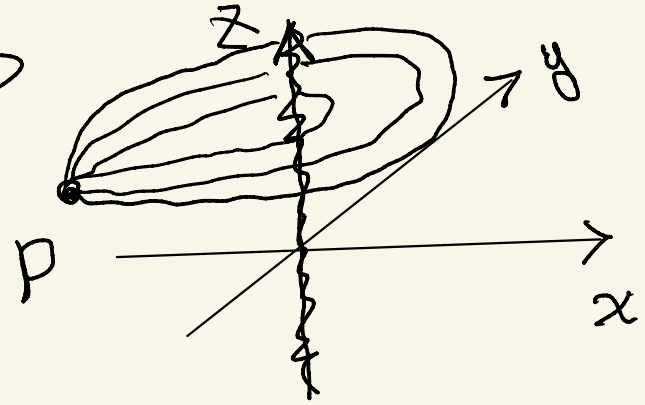
$D = \{\underset{\sim}{x} \in \mathbb{R}^2 : \underset{\sim}{x} \neq 0\}$



Not simply connected because there are closed curves in D which cannot be contracted to a point without passing thru $\underset{\sim}{x} = (0,0)$ and hence "out of D"

Similarly: $\vec{F} = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right)$

$D = \{\underset{\sim}{x} \in \mathbb{R}^3 : x \neq 0, y \neq 0\}$



Check: $\text{Curl}\,\vec{F} = 0$ $\quad \vec{F} = \left(-\frac{y}{r^2}, \frac{x}{r^2}\right)$

$$\text{Curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ M & N & 0 \end{vmatrix} = \hat{i}\cdot 0 - \hat{j}\cdot 0 + (N_x - M_y)\hat{k}$$

$$N_x = \frac{\partial}{\partial x}\frac{x}{r^2} = \frac{1}{r^2} - 2\frac{x}{r^3}\frac{x}{r}$$

$$M_y = -\frac{\partial}{\partial y}\frac{y}{r^2} = -\frac{1}{r^2} + 2\frac{y}{r^3}\frac{y}{r}$$

$$N_x - M_y = \left(\frac{1}{r^2} + \frac{1}{r^2}\right) - 2\frac{1}{r^4}(x^2 + y^2)$$

$\frac{2}{r^2}$ $\qquad$ $-\frac{2}{r^2}$

$$= 0$$

Thus: $\operatorname{curl}\vec{F} = 0$ in D. We show $\int_C \vec{F}\cdot\vec{T}\,ds \neq 0$ on all closed curves. (i.e.) choose

C: unit circle $\quad \vec{r}(t) = \langle \cos t, \sin t \rangle$

$0 \le t \le 2\pi$

$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_0^{2\pi} \vec{F}\cdot\vec{v}\,dt$$

$r = 1$ on unit circle

$$= \int_0^{2\pi} \left\langle \frac{-y}{r^2}, \frac{x}{r^2} \right\rangle \cdot \langle -\sin t, \cos t \rangle\, dt$$

$$= \int_0^{2\pi} \langle -\sin t, \cos t \rangle \cdot \langle -\sin t, \cos t \rangle\, dt$$

$$= \int_0^{2\pi} \sin^2 t + \cos^2 t\, dt = 2\pi$$

$\neq 0$!

Conclude: $\vec{F} = \left(\frac{-y}{r^2}, \frac{x}{r^2}, 0\right)$

is **Curl free** ($\text{Curl}\vec{F} = 0$) on its domain $x \neq 0, y \neq 0$, **but** it can't be conservative because $\int_C \vec{F} \cdot \vec{T}\, ds \neq 0$ for some closed C.

Problem: To conclude $\text{Curl}\vec{F} = 0$ implies $\vec{F} = \nabla f$, we **must have** $\text{Curl}\vec{F} = 0$ on a **simply connected** Domain – the domain being all the values of (x, y, z) where $\text{Curl}\vec{F}$ can be computed, and $\text{Curl}\vec{F}(x, y, z) = 0$.

Example: Consider Newton's Force Field $\vec{F} = -\frac{\vec{r}}{r^3}$

We know its conservation:

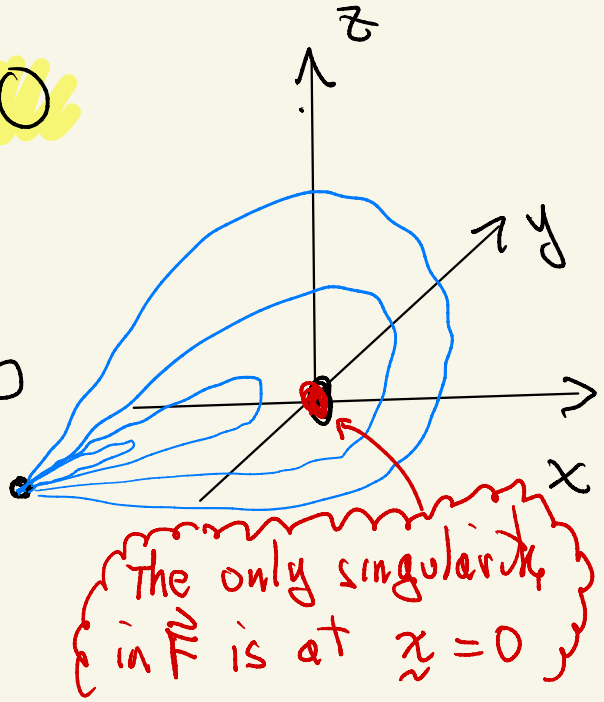
$$\nabla \frac{1}{r} = -\frac{\vec{r}}{r^3} = \vec{F}$$

Q: Is $\vec{F}$ curl free on a simply connected region? Ans Yes

$\vec{F}$ is defined every where

except $\vec{r} = (x,y,z) = 0$

"You can always pull a closed curve around origin in $\mathbb{R}^3$"



Since $\vec{F} = \nabla \frac{1}{r}$ we know

$$\text{Curl}\,\vec{F} = \text{Curl}(\nabla f) = 0$$

because "Curl after ∇ is zero"

$$\mathbb{R} \xrightarrow{\nabla} \mathbb{R}^3 \xrightarrow{\text{Curl}} \mathbb{R}^3 \xrightarrow{\text{Div}} \mathbb{R}$$

We now check directly —

$$\text{Curl}\,\vec{F} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial_x & \partial_y & \partial_z \\ -\frac{x}{r^3} & -\frac{y}{r^3} & -\frac{z}{r^3} \end{vmatrix}$$

$$= \underset{\sim}{i}\left[\left(-\frac{\partial}{\partial y}\left(\frac{z}{r^3}\right) + \frac{\partial}{\partial z}\left(\frac{y}{r^3}\right)\right)\right] - \underset{\sim}{j}\left[-\frac{\partial}{\partial x}\left(\frac{z}{r^3}\right) + \frac{\partial}{\partial z}\left(\frac{x}{r^3}\right)\right]$$

$$+ \underset{\sim}{k}\left[\frac{\partial}{\partial x}\left(-\frac{y}{r^3}\right) - \frac{\partial}{\partial y}\left(-\frac{x}{r^3}\right)\right]$$

$= 0$

Check: $\hat{i}\left[\left(-\frac{\partial}{\partial y}\left(\frac{z}{r^3}\right)+\frac{\partial}{\partial z}\left(\frac{y}{r^3}\right)\right)\right]$

$=0$

$$\frac{\partial}{\partial y}\frac{z}{r^3} = -3\frac{z}{r^4}\frac{y}{r} \qquad \frac{\partial r}{\partial x} = \frac{x}{r} \text{ etc}$$

$$\frac{\partial}{\partial z}\frac{y}{r^3} = -3\frac{y}{r^4}\frac{z}{r}$$

$$-\frac{\partial}{\partial y}\left(\frac{z}{r^3}\right)+\frac{\partial}{\partial z}\left(\frac{y}{r^3}\right) = 3\frac{zy}{r^5}-3\frac{yz}{r^5}=0 \checkmark$$

Similarly for $\hat{j}$ & $\hat{k}$ components.

Conclude: Newton Force $\vec{F} = -\frac{\vec{r}}{r^3}$

Satisfies $\text{Curl}\vec{F}=0$ on a simply connected Domain $\Rightarrow$ Conservative

Thm

Q: Why is it true that if $\text{Curl}\vec{F} = 0$ on a simply connected domain then $\vec{F}$ is conservative?

Ans: Stokes Theorem! I.e. assume so.

Then:

(1) given C closed, it can be contracted to a point to form a surface $\mathcal{S}$ with C as boundary

(2) Stokes thm: $\iint_{\mathcal{S}} \text{Curl}\vec{F} \cdot \vec{n}\, dS = \int_C \vec{F} \cdot \vec{T}\, ds$ ($\text{Curl}\vec{F} = 0$)

(3) Since $\int_C \vec{F} \cdot \vec{T}\, ds = 0$ for every closed curve C, we know line integral is independent of path so $f(x) = \int_A^x \vec{F} \cdot \vec{T}\, ds$ is well defined

(4) $\nabla f = \vec{F}$ by previous Thm, so $\vec{F}$ conservative! ✓